5.1 - Trigonometric Identities

Warmup

Establish the identity

1) $\dfrac{\sin^2(-x) - \cos^2(-x)}{\sin(-x) - \cos(-x)} = \cos x - \sin x$

2) $\dfrac{1 + \tan x}{1 + \cot x} = \tan x$

3) $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = 2 \csc x$

Chapter 5
Trigonometric Identities

1. Trigonometric Identities
2. **Sum and Difference Identities**
3. Double-Angle Identities
4. Half-Angle Identities
5. Product-to-Sum Identities

5.2 - Sum and Difference Identities

Trigonometric functions are not distributive, for example:

$$\cos(A + B) \neq \cos A + \cos B$$

Easy to prove. Let $A = \pi$ and $B = 0$.

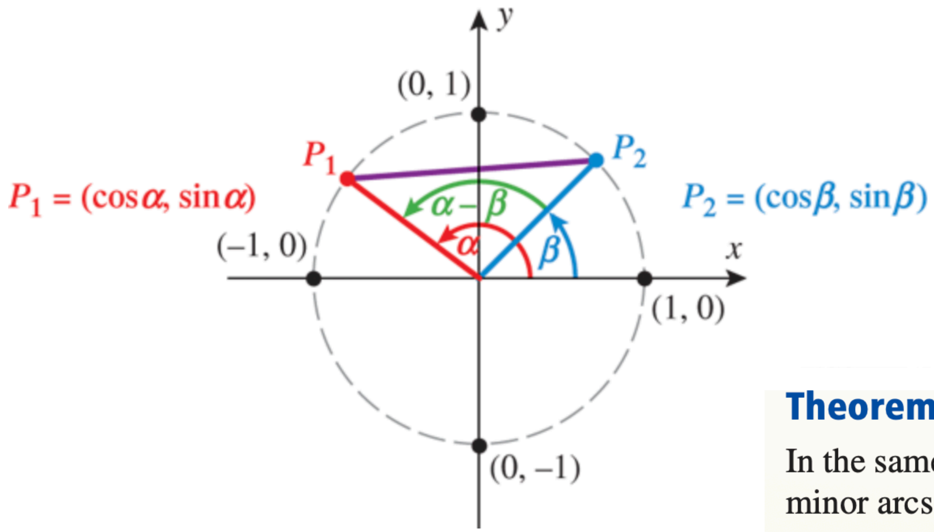
$$\cos(\pi + 0) = \cos \pi = -1$$

Contradiction!!!

$$\cos \pi + \cos 0 = -1 + 1 = 0$$

5.2 - Sum and Difference Identities

Deriving the Cosine difference of angles

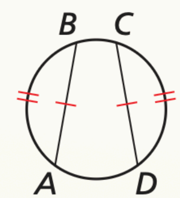


$$P_1 = (\cos\alpha, \sin\alpha)$$

$$P_2 = (\cos\beta, \sin\beta)$$

Theorem 10.6 Congruent Corresponding Chords Theorem

In the same circle, or in congruent circles, two minor arcs are congruent if and only if their corresponding chords are congruent.

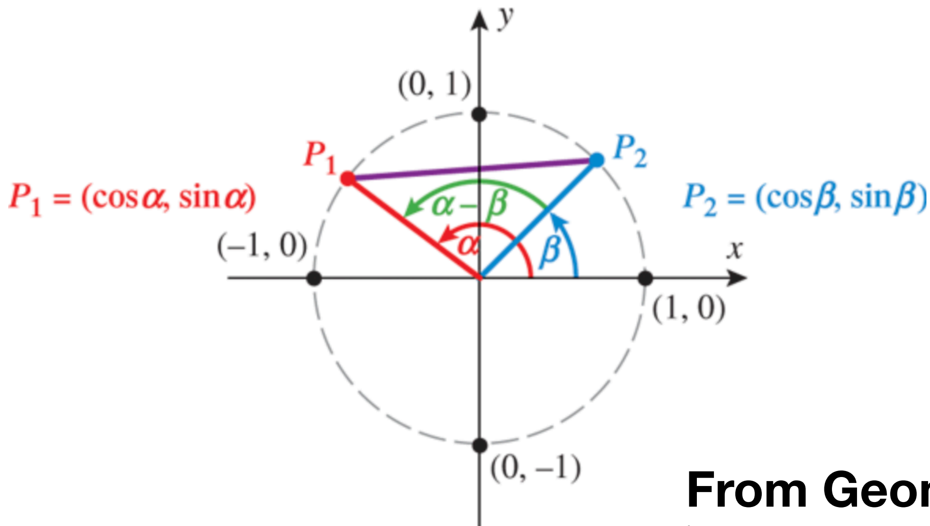


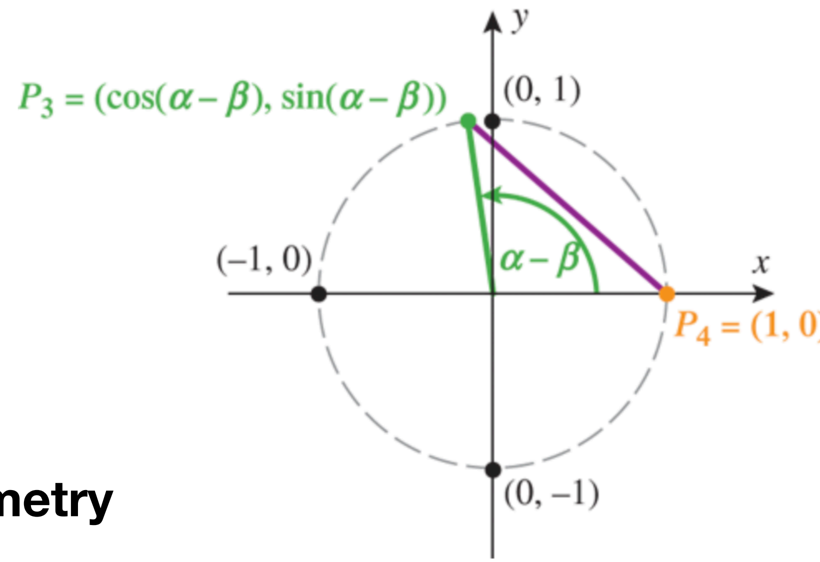
Proof Ex. 19, p. 550

$\overset{\frown}{AB} \cong \overset{\frown}{CD}$ if and only if $\overline{AB} \cong \overline{CD}$.

5.2 - Sum and Difference Identities

Deriving the Cosine difference of angles





From Geometry

$$d_{P_1\ to\ P_2} = d_{P_3\ to\ P_4}$$

$$P_1 = (\cos\alpha, \sin\alpha)$$

$$P_3 = (\cos(\alpha - \beta), \sin(\alpha - \beta))$$

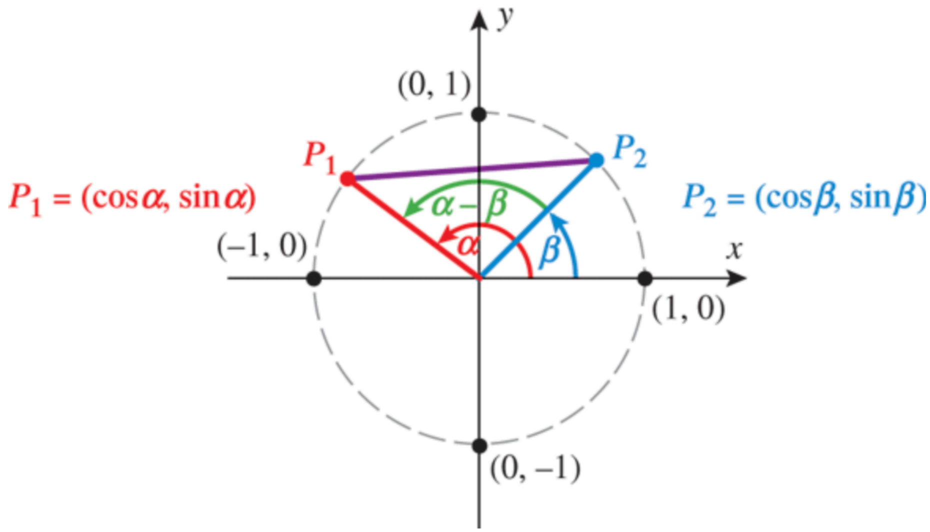
$$P_2 = (\cos\beta, \sin\beta)$$

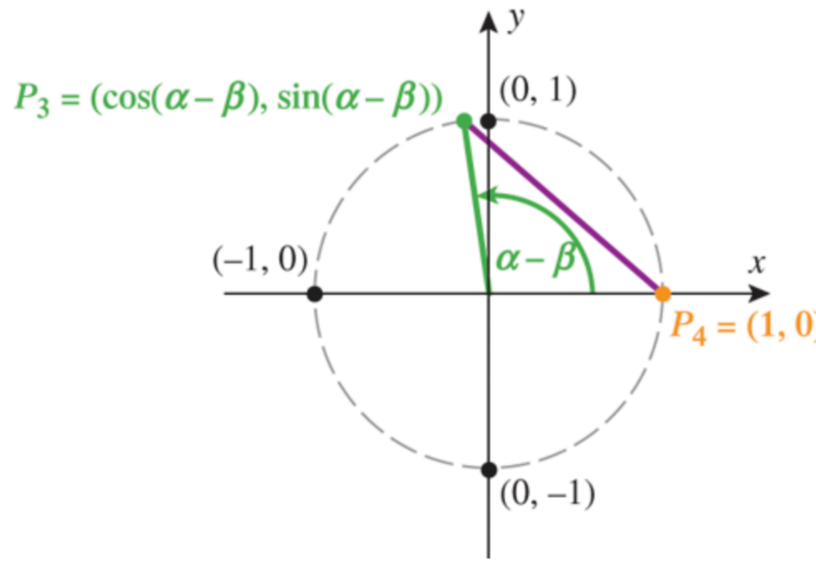
$$P_4 = (1,\ 0)$$

$$\sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2} = \sqrt{(1 - \cos(\alpha - \beta))^2 + (0 - \sin(\alpha - \beta))^2}$$

5.2 - Sum and Difference Identities

Deriving the Cosine difference of angles





$$\sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2} = \sqrt{(1 - \cos(\alpha - \beta))^2 + (0 - \sin(\alpha - \beta))^2}$$

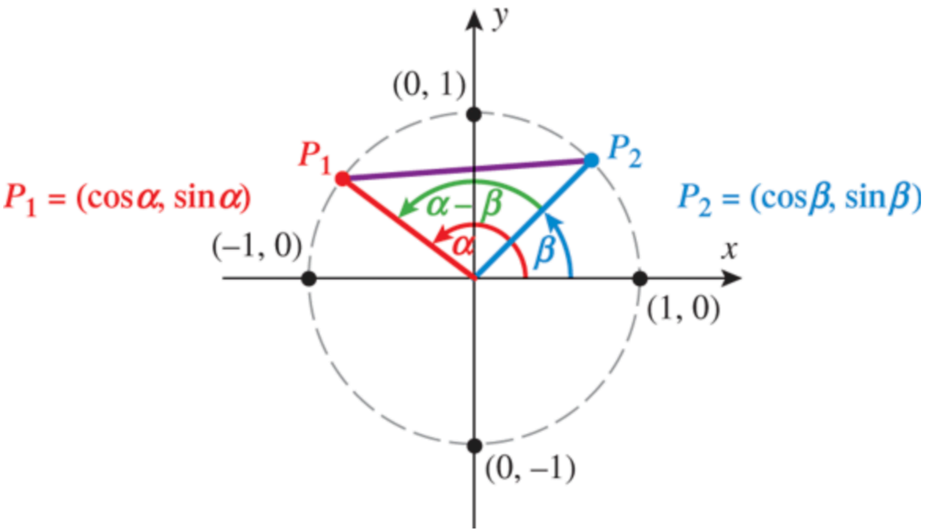
$$(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 = (1 - \cos(\alpha - \beta))^2 + (0 - \sin(\alpha - \beta))^2$$

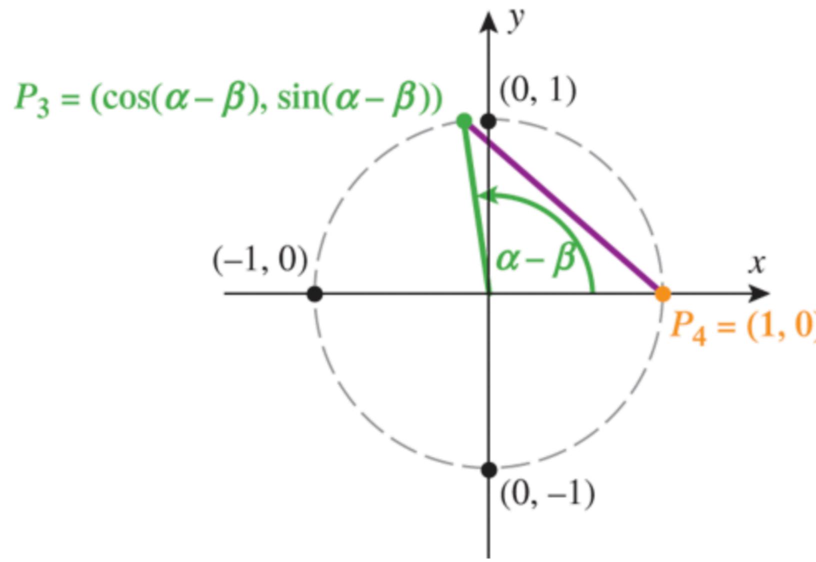
$$\cos^2\beta - 2\cos\alpha\cos\beta + \cos^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta + \sin^2\alpha$$
$$= 1 - 2\cos(\alpha - \beta) + \cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)$$

$$2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos(\alpha - \beta)$$

5.2 - Sum and Difference Identities

Deriving the Cosine difference of angles





$$2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos(\alpha - \beta)$$

$$-2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = -2\cos(\alpha - \beta)$$

$$\boxed{\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)}$$

5.2 - Sum and Difference Identities

Deriving the Cosine sum of angles

Start with the cosine difference identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Substitute β with $-\beta$

$$\cos(\alpha - (-\beta)) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$

$$\boxed{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

5.2 - Sum and Difference Identities

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Evaluate each of the following <u>exactly</u>:

$a.\ \cos 15^\circ$ $\qquad \cos(45^\circ - 30^\circ) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

$b.\ \cos\left(\dfrac{7\pi}{12}\right)$ $\qquad \cos\left(\dfrac{4\pi}{12} + \dfrac{3\pi}{12}\right) = \dfrac{\sqrt{2}-\sqrt{6}}{4}$

$c.\ \cos 75^\circ$ $\qquad \cos(120^\circ - 45^\circ) = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

5.2 - Sum and Difference Identities

Derive Cofunction Identity $\cos(90^\circ - x) = \sin x$

Start with the cosine difference identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Substitute $\alpha = \dfrac{\pi}{2}$ and $\beta = \theta$

$$\cos(\tfrac{\pi}{2} - \theta) = \cos\tfrac{\pi}{2}\cos\theta + \sin\tfrac{\pi}{2}\sin\theta$$

$$\cos(\tfrac{\pi}{2} - \theta) = 0 \cdot \cos\theta + 1 \cdot \sin\theta$$

$$\boxed{\cos(\tfrac{\pi}{2} - \theta) = \sin\theta}$$

5.2 - Sum and Difference Identities

Derive Cofunction Identity $\cos x = \sin(90^\circ - x)$

Again, start with the cosine difference identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Substitute $\alpha = \dfrac{\pi}{2}$ and $\beta = \dfrac{\pi}{2} - \theta$

$$\cos[\tfrac{\pi}{2} - (\tfrac{\pi}{2} - \theta)] = \cos\tfrac{\pi}{2}\cos(\tfrac{\pi}{2} - \theta) + \sin\tfrac{\pi}{2}\sin(\tfrac{\pi}{2} - \theta)$$

$$\cos\theta = 0 \cdot \cos(\tfrac{\pi}{2} - \theta) + 1 \cdot \sin(\tfrac{\pi}{2} - \theta)$$

$$\boxed{\cos\theta = \sin(\tfrac{\pi}{2} - \theta)} \qquad \boxed{\cos(\tfrac{\pi}{2} - \theta) = \sin\theta}$$

5.2 - Sum and Difference Identities

Deriving the Sine sum of angles

Start with the cofunction identity

$$\sin\theta = \cos(\frac{\pi}{2} - \theta)$$

Substitute $\theta = A + B$

$$\sin(A+B) = \cos(\frac{\pi}{2} - (A+B))$$

$$\sin(A+B) = \cos((\frac{\pi}{2} - A) - B)$$

$$\sin(A+B) = \cos(\frac{\pi}{2} - A)\cos B + \sin(\frac{\pi}{2} - A)\sin B$$

$$\boxed{\sin(A+B) = \sin A\cos B + \cos A\sin B}$$

5.2 - Sum and Difference Identities

Deriving the Sine difference of angles

Start with the Sine sum of angles

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Substitute B with $-B$

$$\sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$\boxed{\sin(A - B) = \sin A \cos B - \cos A \sin B}$$

5.2 - Sum and Difference Identities

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Evaluate each of the following exactly:

$a.\ \sin 15^\circ$ $\quad \sin(45^\circ - 30^\circ) = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

$b.\ \sin\left(\dfrac{7\pi}{12}\right)$ $\quad \sin\left(\dfrac{4\pi}{12} + \dfrac{3\pi}{12}\right) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

$c.\ \sin 75^\circ$ $\quad \sin(120^\circ - 45^\circ) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

5.2 - Sum and Difference Identities

Sine/Cosine Identities

Sum/Difference of Angles

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Cofunctions

$$\cos\theta = \sin(\tfrac{\pi}{2} - \theta)$$
$$\sin\theta = \cos(\tfrac{\pi}{2} - \theta)$$

5.2 - Sum and Difference Identities

Deriving the Tangent <u>sum</u> of angles

Start with the Tangent Identity

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$\tan(A+B) = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$\tan(A+B) = \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}}$$

5.2 - Sum and Difference Identities

Deriving the Tangent <u>difference</u> of angles

Start with the Tangent Sum Identity

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Substitute B with $-B$

$$\tan(A + (-B)) = \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$$

$$\boxed{\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

5.2 - Sum and Difference Identities

Sine/Cosine Identities

Sum/Difference of Angles

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Cofunctions

$$\cos\theta = \sin(\tfrac{\pi}{2} - \theta)$$
$$\sin\theta = \cos(\tfrac{\pi}{2} - \theta)$$

Tangent Identities

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

5.2 - Sum and Difference Identities

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}\qquad \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

Practice - Establish the identity

$$a.\ \frac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta}=\cot\alpha\cot\beta+1$$

$$b.\ \tan\left(\theta+\frac{\pi}{2}\right)=-\cot\theta$$

Use

$$\tan(\theta+\tfrac{\pi}{2})=\frac{\sin(\theta+\pi/2)}{\cos(\theta+\pi/2)}$$

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